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PROPERTY MATRICES IDENTIFICATION OF UNBOUNDED MEDIUM FROM UNIT-IMPULSE RESPONSE FUNCTIONS USING LEGENDRE POLYNOMIALS: FORMULATION

ANTONIO PARONESSO AND JOHN P. WOLF

Department of Civil Engineering, Institute of Hydraulics and Energy, Swiss Federal Institute of Technology Lausanne, CH-1015 Lausanne, Switzerland

SUMMARY

A systematic procedure to construct the (symmetric) static-stiffness, damping and mass matrices representing the unbounded medium is presented addressing the unit-impulse response matrix corresponding to the degrees of freedom on the structure–medium interface. The unit-impulse response matrix is first diagonalized which then permits each term to be modelled independently from the others using expansions in a series of Legendre polynomials in the time domain. This leads to a rational approximation in the frequency domain of the dynamic-stiffness coefficient. Using a lumped-parameter model which provides physical insight the property matrices are constructed.

KEY WORDS: Legendre polynomials; property matrices identification; rational approximation; soil–structure interaction; unbounded-medium–structure interaction; unit-impulse response

1. INTRODUCTION

To analyse the dynamic interaction of a structure (e.g. a building or a dam) with the adjacent unbounded (semi-infinite) medium (soil, reservoir of water) based on the substructure method, the two substructures are coupled on the structure–medium interface. The modelling of the bounded non-linear structure with finite elements is well understood resulting in the banded static-stiffness, damping and mass matrices called the property matrices, corresponding to a finite number of degrees of freedom. The representation of the unbounded linear medium, although much more difficult, is also possible, introducing the *unit-impulse response matrix* $[S(t)]$. The interaction force $\{R(t)\}$–displacement $\{u(t)\}$ relationship with respect to the degrees of freedom of the nodes on the structure–medium interface of the unbounded medium (see for instance Chapter 6 of Reference 1) is global in space and time (Figure 1)

$$\{R(t)\} = \int_0^t [S(t-\tau)]\{u(\tau)\}\,\mathrm{d}\tau = [K_\infty]\{u(t)\} + [C_\infty]\{\dot{u}(t)\} + \{R_r(t)\} \tag{1}$$

The first two terms on the right-hand side representing the instantaneous response define the singular part with $[K_\infty]$ and $[C_\infty]$ denoting the high-frequency limit ($\omega \to \infty$) of the dynamic-stiffness matrix $[S(\omega)]$. The third term describing the lingering response is equal to the regular part (subscript r) consisting of the convolution integral of the corresponding unit-impulse response matrix $[S_r(t)]$ and the displacement vector

$$\{R_r(t)\} = \int_0^t [S_r(t-\tau)]\{u(\tau)\}\,\mathrm{d}\tau \tag{2}$$

The interaction forces at a specific time depend on the time histories of the displacements in all nodes from the start of the excitation onwards. In this rigorous formulation a large computational effort (proportional to the square of the number of the time stations) and storage requirement result, which makes it unrealistic to evaluate large practical problems such as e.g. the seismic analysis of a dam with many degrees of freedom on the dam–soil and dam–reservoir interfaces.

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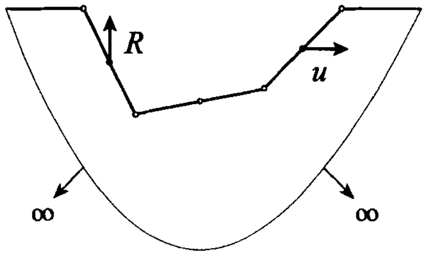

Figure 1. Interaction force–displacement relationship on structure–medium interface of unbounded medium

In the frequency domain, the corresponding relationship formulated in amplitudes equals

$$\{R(\omega)\} = [S(\omega)]\{u(\omega)\} = ([K_\infty] + i\omega[C_\infty])\{u(\omega)\} + \{R_r(\omega)\} \tag{3}$$

where $\lim_{\omega\to\infty} [S(\omega)] = [K_\infty] + i\omega[C_\infty]$ and

$$\{R_r(\omega)\} = [S_r(\omega)]\{u(\omega)\} \tag{4}$$

Sophisticated reliable methods to calculate $[K_\infty]$, $[C_\infty]$ and $[S_r(t_j)]$ at distinct time stations t_j based on similarity using the finite-element method exist,[2–4] which can also be used to calculate $[S_r(\omega_j)]$ at distinct frequencies ω_j.[5] Alternatively, the boundary-element method can be applied.[6,7]

The spatial discretization of the structure–medium interface leads to a limiting frequency ω_{max}, up to which the accuracy of $[S_r(\omega_j)]$ is acceptable ($\omega_j \leqslant \omega_{max}$). This corresponds in the time domain to a maximum time-step Δt_{max} which can be used in the calculation of $[S_r(t_j)]$ ($\Delta t \leqslant \Delta t_{max}$). For computational efficiency, the number of time steps must be limited, which defines the maximum time t_{max} up to which $[S_r(t_j)]$ is calculated ($t_j \leqslant t_{max}$). Thus, only partial data are available.

To reduce the computational effort of the evaluation of the interaction forces during an actual analysis, concepts of *linear system theory* can be applied.[8,9] These consist of introducing a *rational approximation of the dynamic-stiffness matrix* in the frequency domain $[S(\omega)]$, i.e. each coefficient is a ratio of two polynomials in $i\omega$ with real coefficients. This corresponds for the continuous time system (as the so-called realization in the time domain) to a system of ordinary linear first-order differential equations with constant coefficients which also introduce additional internal variables

$$\{\dot{x}(t)\} = [A]\{x(t)\} + [B]\{u(t)\} \tag{5a}$$

$$\{R(t)\} = [C]\{x(t)\} + [K_\infty]\{u(t)\} + [C_\infty]\{\dot{u}(t)\} \tag{5b}$$

with the state-variables (internal variables) $\{x(t)\}$ with M elements and the time-independent matrices $[A]$, $[B]$, $[C]$, which have to be determined. Equation (5) can also be written as

$$\begin{bmatrix} [I] & [0] \\ [0] & [C_\infty] \end{bmatrix} \begin{Bmatrix} \{\dot{x}(t)\} \\ \{\dot{u}(t)\} \end{Bmatrix} + \begin{bmatrix} -[A] & -[B] \\ [C] & [K_\infty] \end{bmatrix} \begin{Bmatrix} \{x(t)\} \\ \{u(t)\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{R(t)\} \end{Bmatrix} \tag{6}$$

Note that the coefficient matrix of $\{x(t)\}$ and $\{u(t)\}$ in the second term on the left-hand side is, in general, not symmetric.

A recent literature review on rational approximation and realization addressing the dynamic analysis of an unbounded medium is specified in Reference 10 and will not be repeated here.

In Reference 10, which addresses the continuous-time system, the starting point is the dynamic-stiffness matrix $[S(\omega)]$ for $\omega \leqslant \omega_{max}$. The subscript j is dropped in ω_j for conciseness. It is demonstrated that it is efficient to first diagonalize $[S(\omega)]$ using a transformation without introducing any approximation. This results in a diagonal matrix $[S^m(\omega)]$ (superscript m for modelled). The rational approximation and realization are then performed for each diagonal element $S^m(\omega)$ independently from the others, directly in the frequency domain, leading to $S^m(i\omega)$. Using a local lumped-parameter model the (symmetric) static-stiffness

and damping matrices of each element $S^{m}(i\omega)$ follow without introducing any additional approximation corresponding to a system of first-order differential equations. Alternatively, a mass matrix can be introduced yielding a system of second-order differential equations. After assemblage and backtransformation the global system of first- and second-order differential equations are formulated as

$$[C^1]\begin{Bmatrix}\{\dot{x}(t)\}\\ \{\dot{u}(t)\}\end{Bmatrix} + [K^1]\begin{Bmatrix}\{x(t)\}\\ \{u(t)\}\end{Bmatrix} = \begin{Bmatrix}\{0\}\\ \{R(t)\}\end{Bmatrix} \tag{7}$$

$$[M^2]\begin{Bmatrix}\{\ddot{w}(t)\}\\ \{\ddot{u}(t)\}\end{Bmatrix} + [C^2]\begin{Bmatrix}\{\dot{w}(t)\}\\ \{\dot{u}(t)\}\end{Bmatrix} + [K^2]\begin{Bmatrix}\{w(t)\}\\ \{u(t)\}\end{Bmatrix} = \begin{Bmatrix}\{0\}\\ \{R(t)\}\end{Bmatrix} \tag{8}$$

For instance, $[M^2]$, $[C^2]$ and $[K^2]$ are the global mass, damping and static-stiffness matrices of the realization of the unbounded medium (superscript 2 for second-order realization). $\{u(t)\}$ is the vector of displacements on the structure–medium interface and $\{w(t)\}$ is the vector of the internal degrees of freedom of the second-order realization. Its order can be one half of that of the internal degrees of freedom $\{x(t)\}$ of the first-order realization (state-variables). The symmetric forms of equations (7) and (8) are equivalent to the non-symmetric equation (6), i.e. the corresponding dynamic-stiffness matrices relating $\{u(\omega)\}$ to $\{R(\omega)\}$ are identical. In this formulation the unbounded medium is represented in the same manner as the bounded structure, and thus the same computer programme can be applied. After assemblage with the property matrices of the structure, the resulting equations of motions can be solved using a time integrator. The number of operations is proportional to the number of time steps. In Reference 10, a consistent global lumped-parameter model connecting the degrees of freedom on the structure–medium interface for the unbounded medium is also derived.

It is the goal of the present paper to develop an analogous formulation using as starting point the regular part of the unit-impulse response matrix $[S_r(t_j)]$ for $t_j \leqslant t_{max}$. Elements of *linear system identification* are applied to construct a rational approximation of the dynamic-stiffness matrix in the frequency domain. In this *continuous time-system* formulation, a diagonalization is again performed without introducing any approximation, but for $[S_r(t_j)]$ (Section 2). For each resulting element $S_r^m(t_j)$ the suitably chosen input and the corresponding output are expanded into a series of Legendre polynomials which lead to the coefficient matrix of a linear overdeterminated system. The least-squares solution determines the coefficients of a linear differential equation whose unit-impulse response represents an approximation of $\tilde{S}_r^m(t_j)$ with the corresponding $S_r^m(i\omega)$ representing a rational function in $i\omega$ (Section 3). The realization then follows without introducing any further approximation yielding the static-stiffness, damping and possibly mass matrices as before (Section 4).

The optimum implementation and examples are addressed in the accompanying paper.[11]

2. DIAGONALIZATION

In principle, the methods of linear system theory can be applied directly to the total matrices $[S(\omega)]$ or $[S(t)]$. The subscript j in ω_j and t_j is dropped. Working directly with the total matrix reduces the number of internal variables and thus the storage requirement, but the computational effort per time is reduced less compared to processing each element of the matrix independently from the others.[12] This is caused by the fact that in the formulation working with the total matrix, fully coupled submatrices linking the internal variables to the degrees of freedom on the structure–medium interface arise in the realization, while in the element-by-element procedure the matrices corresponding to the total system are sparse. As the number of degrees of freedom in the nodes on the structure–medium interface is, in general, quite large, the involved huge matrices when addressing the total matrix become unmanageable when performing the rational approximation. Thus, in a typical unbounded medium–structure interaction analysis a procedure should be used which addresses each element of the matrix separately. In addition, this diagonalization procedure makes use of the symmetry of the matrix reducing the computational effort of the actual calculation in the time domain. As another advantage compared to processing each element of the matrix separately from the others, the property matrices of the realization are symmetric.

The diagonalization procedure described in Reference 10 in the frequency domain working with $[S(\omega)]$ is directly applicable in the time domain addressing $[S(t)]$. The diagonalization involving a transformation can be formulated for $[S(t)]$ (equation (1)) or for $[S_r(t)]$ (equation (2)), as in the latter case the singular part described by $[K_\infty]$ and $[C_\infty]$ is already in a symmetric rational form. The diagonalization is summarized for $[S_r(t)]$. Reference 10 should also be consulted.

The number of degrees of freedom in the nodes on the structure–medium interface (Figure 1) is denoted as N. The regular part of the interaction forces $\{R_r(t)\}$ is formulated as (equation (2))

$$\{R_r(t)\} = \int_0^t [S_r(t-\tau)]\{u(\tau)\}\,\mathrm{d}\tau \tag{9}$$

The regular part of the (symmetric) unit-impulse response matrix of the unbounded medium $[S_r(t)]$ of order $N \times N$ is transformed to an equivalent diagonal matrix $[S_r^{\mathrm{m}}(t)]$ with $N(N+1)/2$ terms which contains the modelled dynamic-stiffness coefficients. No approximation is introduced. With the time-independent transformation matrix $[T]$, the relationship

$$[S_r(t)] = [T][S_r^{\mathrm{m}}(t)][T]^{\mathrm{T}} \tag{10}$$

is formulated. The construction of $[T]$ is discussed later. This corresponds to the following transformation

$$\{u^{\mathrm{m}}(t)\} = [T]^{\mathrm{T}}\{u(t)\}, \qquad \{R_r(t)\} = [T]\{R_r^{\mathrm{m}}(t)\} \tag{11a, b}$$

where $\{u^{\mathrm{m}}(t)\}$ and $\{R_r^{\mathrm{m}}(t)\}$ are the input and output vectors of order $N(N+1)/2$ related by

$$\{R_r^{\mathrm{m}}(t)\} = \int_0^t [S_r^{\mathrm{m}}(t-\tau)]\{u^{\mathrm{m}}(\tau)\}\,\mathrm{d}\tau \tag{12}$$

Equation (10) can also be written as

$$[E]\{S_r^{\mathrm{m}}(t)\} = \{S_r(t)\} \tag{13}$$

$\{S_r^{\mathrm{m}}(t)\}$ contains the $N(N+1)/2$ unknown time-dependent coefficients of $[S_r^{\mathrm{m}}(t)]$ and $\{S_r(t)\}$ the $N(N+1)/2$ known independent coefficients of the upper triangular submatrix of $[S_r(t)]$ ordered row-wise. $[E]$ is calculated from $[T]$ as follows. The kth row of $[E]$ corresponding to the kth element of the right-hand side $S_{rk}(t) = S_{rij}(t)$ consists of the elements

$$E_{kh} = T_{ih}T_{jh} \quad (k, h = 1, 2, \ldots, N(N+1)/2) \tag{14}$$

Solution of equation (13) yields $\{S_r^{\mathrm{m}}(t)\}$ and thus $[S_r^{\mathrm{m}}(t)]$.

The construction of the $[T]$ matrix is discussed. Various possibilities exist, of which two are addressed. As explained in Reference 10, the $[T]$ matrix is chosen as the transpose of the kinematic matrix linking the degrees of freedom in the nodes to the distortions of a suitably chosen global lumped-parameter model with rigid bars. This first procedure which is not examined in detail, permits a physical interpretation of the final static-stiffness, damping and mass matrices of the realization which corresponds to the global lumped-parameter model with time independent coefficients connecting the degrees of freedom of the structure–medium interface. When only the property matrices are of interest, a more straightforward construction of the $[T]^{\mathrm{T}}$ matrix is possible.

In this second procedure, N rows exist for which the only non-zero element equals $+1$. In all other rows, one element $+1$ and one element -1 appear with the remaining elements vanishing. To construct the transpose of the $[T]$ matrix with the elements $TT(I, J)$ $(I = 1, \ldots, N(N+1)/2, J = 1, \ldots, N)$ for a system with N degrees of freedom the following algorithm described in the flowchart of Figure 2 applies. The matrix can be subdivided into N blocks with the index K. The first block has N rows, the second block $N-1$ rows, and the Nth block one row.

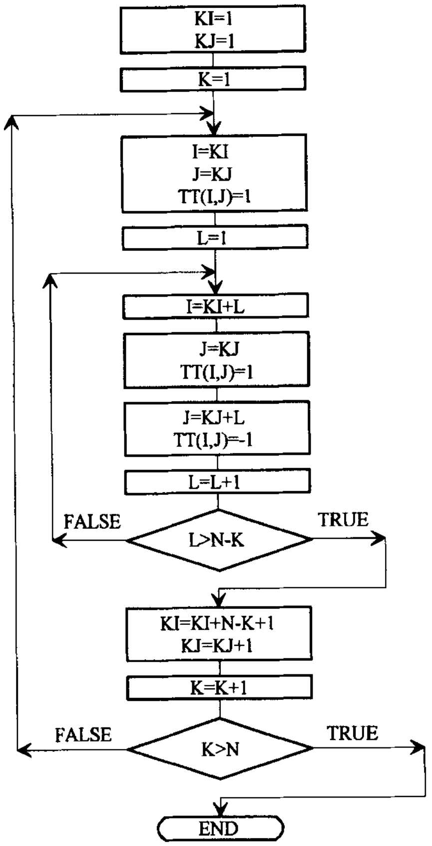

Figure 2. Flowchart to construct the transpose of the transformation matrix $[T]$

As an example, $N = 3$ results in

$$[T]^{\mathrm{T}} = \begin{bmatrix} 1 & & \\ 1 & -1 & \\ 1 & & -1 \\ \cdots & \cdots & \cdots \\ & 1 & \\ & 1 & -1 \\ \cdots & \cdots & \cdots \\ & & 1 \end{bmatrix} \tag{15}$$

Using the $[T]$ matrix constructed according to Figure 2, equations (14) and (13) lead to an explicit expression for the elements in $\{S_r^{\mathrm{m}}(t)\}$ which depend on $\{S_r(t)\}$ and thus on the original matrix $[S_r(t)]$. To each element $S_{rk}^{\mathrm{m}}(t)$ with the index k in $\{S_r^{\mathrm{m}}(t)\}$ corresponds an element $S_{rk}(t)$ with the same index k in $\{S_r(t)\}$. If this $S_{rk}(t)$ corresponds to a diagonal element $S_{rii}(t)$ of $[S(t)]$

$$S_{rk}^{\mathrm{m}}(t) = \sum_{l=1}^{N} S_{ril}(t) \tag{16}$$

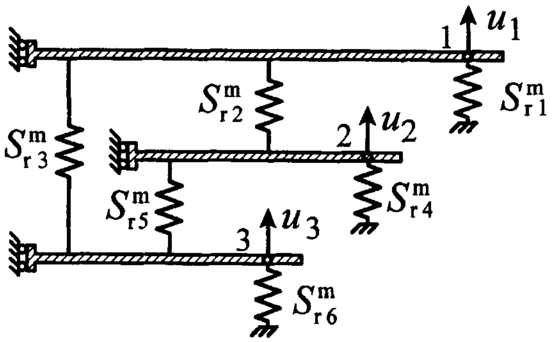

Figure 3. Diagonalization procedure interpreted physically as a global lumped-parameter model where the displacements in the nodes are parallel

applies. Otherwise, i.e. if this $S_{rk}(t)$ corresponds to an off-diagonal element $S_{ril}(t)$ $(i \neq l)$ of $[S(t)]$

$$S^{\mathrm{m}}_{rk}(t) = -S_{ril}(t) \tag{17}$$

holds.

For the example $N = 3$, equations (16) and (17) lead to

$$S^{\mathrm{m}}_{r1}(t) = S_{r11}(t) + S_{r12}(t) + S_{r13}(t), \qquad S^{\mathrm{m}}_{r2}(t) = -S_{r12}(t), \qquad S^{\mathrm{m}}_{r3}(t) = -S_{r13}(t) \tag{18a, b, c}$$

$$S^{\mathrm{m}}_{r4}(t) = S_{r21}(t) + S_{r22}(t) + S_{r23}(t), \qquad S^{\mathrm{m}}_{r5}(t) = -S_{r23}(t) \tag{18d, e}$$

$$S^{\mathrm{m}}_{r6}(t) = S_{r31}(t) + S_{r32}(t) + S_{r33}(t) \tag{18f}$$

The $[T]$ matrix constructed according to the algorithm of Figure 2 corresponds to the transpose of the kinematic matrix of a global lumped-parameter model with generalized springs where the displacements in the nodes are all parallel. In this case the regular part of the unit-impulse response coefficients in equations (16) and (17) correspond to those of the generalized springs in the lumped-parameter model.

For the example $N = 3$ with parallel displacements in the nodes u_1, u_2, u_3 the lumped-parameter model is illustrated in Figure 3. Its regular parts of the unit-impulse response coefficients are specified in equation (18). They can be formulated directly based on physical grounds. Applying e.g. $u_1(t) = u_2(t) = u_3(t) = 1$ to the global lumped-parameter model yields as the external force applied in node 1 $S^{\mathrm{m}}_{r1}(t)$ which is equal to $S_{r11}(t) + S_{r12}(t) + S_{r13}(t)$ (equation (18a)).

In the result of this diagonalization expressed in the input–output relationship of equation (12), each term $S^{\mathrm{m}}_r(t)$ can be addressed independently from the others. The rational approximation is discussed in Section 3 and the realization in Section 4 for the (scalar) term $S^{\mathrm{m}}_r(t)$.

Often, in a practical application, a reduction of the number of degrees of freedom on the structure–medium interface of the unbounded medium is performed. This is achieved using a time-independent transformation matrix whose columns represent Ritz vectors. The interaction force–displacement relationship of equations (1) and (2) is then formulated defining the regular part of the unit-impulse response matrix which is the starting point of the diagonalization procedure. The two transformation matrices of the Ritz vector reduction and of the diagonalization can be combined in an actual analysis working with the product of these two matrices.

3. SYSTEM IDENTIFICATION USING LEGENDRE POLYNOMIALS

The input–output relationship with the diagonal matrix in equation (12) is formulated for each term as

$$R^{\mathrm{m}}_r(t) = \int_0^t S^{\mathrm{m}}_r(t-\tau)\, u^{\mathrm{m}}(\tau)\, \mathrm{d}\tau \tag{19}$$

$S_r^m(t)$ represent the regular part of the unit-impulse response coefficient $S^m(t)$ which is known at discrete times $t_j \leqslant t_{max}$. Piecewise linear interpolation is applied yielding $S_r^m(t)$. Note that $S_r^m(t)$ is available only for $t \leqslant t_{max}$. $u^m(t)$ denotes the transformed displacement and $R_r^m(t)$ the regular part of the transformed interaction force. $u^m(t)$ and $R_r^m(t)$ represent the input and the output, respectively. In the frequency domain the corresponding equation equals

$$R_r^m(\omega) = S_r^m(\omega) u^m(\omega) \tag{20}$$

where $S_r^m(\omega)$ is the Fourier transform of $S_r^m(t)$.

To construct a rational function in the frequency domain representing an approximation $S_r^m(i\omega)$ for the dynamic-stiffness coefficient $S_r^m(\omega)$, concepts of linear system identification are applied. For a chosen input $u^m(t)$ ($t \leqslant t_{max}$), the output $R_r^m(t)$ ($t \leqslant t_{max}$) can be calculated by evaluating the convolution integral of equation (19). An input–output pair is thus available for $t \leqslant t_{max}$. This is the starting point in system identification where it is customary to measure the output for a given input. This theory leads to a linear dynamic system described by a finite number of parameters. The corresponding dynamic-stiffness coefficient will be a rational function. As will be explained in more detail, both $u^m(t)$ and $R_r^m(t)$ are expanded in a series of Legendre polynomials. The coefficients of this series permit the unknown coefficients of the rational function to be determined.

This determination of the parameters using Legendre polynomials as used in system identification is described in References 13–15. In the following summary only those equations necessary for the implementation are addressed.

Starting from the basic polynomials $1, t, t^2, \ldots$ and applying orthogonalization for $t \leqslant t_{max}$ yield the set of (shifted) Legendre polynomials $\varphi_i(t)$ ($i = 0, 1, 2, \ldots$). They can be constructed for $t \leqslant t_{max}$ recursively using

$$\varphi_0(t) = 1, \qquad \varphi_1(t) = 2\frac{t}{t_{max}} - 1 \tag{21a, b}$$

$$\varphi_{i+1}(t) = \frac{2i+1}{i+1}\left(2\frac{t}{t_{max}} - 1\right)\varphi_i(t) - \frac{i}{i+1}\varphi_{i-1}(t) \quad (i \geqslant 1) \tag{21c}$$

The Legendre polynomials form a complete set. They are orthogonal

$$\int_0^{t_{max}} \varphi_i(t)\varphi_j(t)\,dt = \frac{t_{max}}{2i+1}\delta_{ij} \tag{22}$$

where δ_{ij} is the Kroneker delta ($\delta_{ij} = 1$ for $i = j$ and $= 0$ for $i \neq j$). In mathematical terms, this set is complete in the normed linear space $L_2[0, t_{max}]$ of the square integrable functions over the interval $0 \leqslant t \leqslant t_{max}$. Any function $f(t)$ which is square integrable over the interval $0 \leqslant t \leqslant t_{max}$ can be expanded in a Legendre series with l terms

$$f(t) \cong \sum_{i=0}^{l-1} c_i \varphi_i(t) \tag{23}$$

Multiplying equation (23) by $\varphi_j(t)$, integrating over $0 \leqslant t \leqslant t_{max}$ and using equation (22) yields the coefficients

$$c_i = \frac{2i+1}{t_{max}} \int_0^{t_{max}} f(t)\varphi_i(t)\,dt \tag{24}$$

It can be shown that, due to the property of orthogonality, the polynomial of degree $l - 1$ on the right-hand side of equation (23) represents the best approximation in the mean of all polynomials of the same degree

$$\varepsilon_{l-1} = \int_0^{t_{max}} \left| f(t) - \sum_{i=0}^{l-1} c_i \varphi_i(t) \right|^2 dt \tag{25}$$

Due to the completeness property the error ε_{l-1} tends to zero for $l \to \infty$, resulting in convergence in the mean of the series. Placing the functions $\varphi_i(t)$ in a vector $\{\varphi(t)\}$ of order l and c_i in $\{c\}$ equation (23) is formulated as

$$f(t) \cong \{c\}^T \{\varphi(t)\} \tag{26}$$

As will become apparent, the integral of $\{\varphi(t)\}$ from 0 to t has to be evaluated. The integral can be written as

$$\int_0^t \{\varphi(t)\}\,\mathrm{d}t = [[L]\ \{L_l\}]\begin{Bmatrix}\{\varphi(t)\}\\ \varphi_l(t)\end{Bmatrix} \tag{27}$$

with the so-called operational matrix of integration for the Legendre polynomials $[L]$ of order $l\times l$

$$[L] = \begin{bmatrix} 1/2 & 1/2 & & & \cdots & & \\ -1/6 & & 1/6 & & \cdots & & \\ & -1/10 & & 1/10 & \cdots & & \\ \vdots & \vdots & \vdots & \ddots & \cdots & & \\ & & & \dfrac{-1}{2(2l-3)} & & & \dfrac{1}{2(2l-3)} \\ & & & & \dfrac{-1}{2(2l-1)} & & \end{bmatrix}, \quad \{L_l\} = \begin{Bmatrix} \\ \\ \\ \vdots \\ \\ \dfrac{1}{2(2l-1)} \end{Bmatrix} \tag{28,29}$$

Equation (27) is derived based on the properties of the Legendre polynomials. It can be verified straightforwardly through integration. From the last row of equation (27) it follows that the integration of a Legendre polynomial of degree $l-1$ results in a linear combination of Legendre polynomials of degrees $l-2$ and l with the coefficients $-1/(2(2l-1))$ and $1/(2(2l-1))$, respectively (last lines of equations (28) and (29)). In the derivation, $\{L_l\}$ is suppressed, yielding from equation (27)

$$\int_0^t \{\varphi(t)\}\,\mathrm{d}t \cong [L]\{\varphi(t)\} \tag{30}$$

For a large l the neglected term $1/(2(2l-1))$ in equation (29) tends to zero. Integrating equation (30) k times yields

$$\underbrace{\int_0^t}_{k\ \text{times}} \{\varphi(t)\}\,\mathrm{d}t \cong [L]^k\{\varphi(t)\} \tag{31}$$

The regular part of the unit-impulse response coefficient $S_r^{\mathrm{m}}(t)$ in the input–output relationship of equation (19) is approximated as that corresponding to an ordinary differential equation of order M for the output $R_r^{\mathrm{m}}(t)$, whereby derivatives up to $M-1$ for the input $u^{\mathrm{m}}(t)$ are present

$$q_0 R_r^{\mathrm{m}}(t) + q_1\frac{\mathrm{d}R_r^{\mathrm{m}}(t)}{\mathrm{d}t} + q_2\frac{\mathrm{d}^2R_r^{\mathrm{m}}(t)}{\mathrm{d}t^2} + \cdots + \frac{\mathrm{d}^M R_r^{\mathrm{m}}(t)}{\mathrm{d}t^M}$$

$$= p_0 u^{\mathrm{m}}(t) + p_1\frac{\mathrm{d}u^{\mathrm{m}}(t)}{\mathrm{d}t} + p_2\frac{\mathrm{d}^2u^{\mathrm{m}}(t)}{\mathrm{d}t^2} + \cdots + p_{M-1}\frac{\mathrm{d}^{M-1}u^{\mathrm{m}}(t)}{\mathrm{d}t^{M-1}} \tag{32}$$

All $2M$ unknown coefficients $q_0, \ldots, p_{M-1}$ are constant and real. Note that the coefficient of $\mathrm{d}^M R_r^{\mathrm{m}}(t)/\mathrm{d}t^M$ is selected as one. In the algorithm the order M must be chosen.

The Fourier transformation of equation (32) leads to the input–output relationship in the frequency domain

$$R_r^{\mathrm{m}}(\omega) = S_r^{\mathrm{m}}(\mathrm{i}\omega)u^{\mathrm{m}}(\omega) \tag{33}$$

where the approximated regular part of the dynamic-stiffness coefficient equals

$$S_r^{\mathrm{m}}(\mathrm{i}\omega) = \frac{p_0 + p_1(\mathrm{i}\omega) + p_2(\mathrm{i}\omega)^2 + \cdots + p_{M-1}(\mathrm{i}\omega)^{M-1}}{q_0 + q_1(\mathrm{i}\omega) + q_2(\mathrm{i}\omega)^2 + \cdots + (\mathrm{i}\omega)^M} \tag{34}$$

$S_r^{\mathrm{m}}(\mathrm{i}\omega)$ is a rational function in $\mathrm{i}\omega$ with the coefficients $q_0, \ldots, p_{M-1}$ where the degrees of the polynomials in the denominator and the numerator are equal to M and $M-1$, respectively. Note that the argument of the rational approximation $S_r^{\mathrm{m}}(\mathrm{i}\omega)$ is denoted as $\mathrm{i}\omega$ in contrast to that of the original $S_r^{\mathrm{m}}(\omega)$. For the limit of $\mathrm{i}\omega \to \infty$ the approximation of the regular part tends to zero. The approximate dynamic-stiffness coefficient is thus exact in the high-frequency limit (asymptotic behaviour).

The rational function in equation (34) can be expressed as a partial-fraction expansion of the form

$$S_r^{\mathrm{m}}(\mathrm{i}\omega) = \sum_{j=1}^{M} \frac{A_j}{\mathrm{i}\omega - s_j} \tag{35}$$

where s_j are the roots of the polynomial in the denominator, i.e. the poles of $S_r^{\mathrm{m}}(\mathrm{i}\omega)$, and the A_j are the residues at the poles

$$A_j = (\mathrm{i}\omega - s_j) S_r^{\mathrm{m}}(\mathrm{i}\omega)\big|_{\mathrm{i}\omega = s_j} \tag{36}$$

Equation (35) is valid for first-order poles, which is the case in practical applications.

The selection of the input $u^{\mathrm{m}}(t)$ is discussed in the accompanying paper.[11] For a specified $u^{\mathrm{m}}(t)$, the output $R_r^{\mathrm{m}}(t)$ is calculated by evaluating the convolution integral in equation (19). Both $u^{\mathrm{m}}(t)$ and $R_r^{\mathrm{m}}(t)$ are then expanded in a Legendre series with l terms (equation (26))

$$u^{\mathrm{m}}(t) \cong \{c_u\}^{\mathrm{T}} \{\varphi(t)\} \tag{37}$$

$$R_r^{\mathrm{m}}(t) \cong \{c_R\}^{\mathrm{T}} \{\varphi(t)\} \tag{38}$$

with the coefficients c_{ui} and c_{Ri} of $\{c_u\}$ and $\{c_R\}$ determined as (equation (24))

$$c_{ui} = \frac{2i+1}{t_{\max}} \int_0^{t_{\max}} u^{\mathrm{m}}(t)\varphi_i(t)\,\mathrm{d}t \tag{39}$$

$$c_{Ri} = \frac{2i+1}{t_{\max}} \int_0^{t_{\max}} R_r^{\mathrm{m}}(t)\varphi_i(t)\,\mathrm{d}t \tag{40}$$

To determine the coefficients $q_0, \ldots, p_{M-1}$, equation (32) is integrated M times, which transforms the differential equation of Mth order to an integral equation. The system which is assumed to be initially at rest satisfies the following initial conditions at $t = 0^-$

$$\frac{\mathrm{d}^{M-2}u^{\mathrm{m}}(t)}{\mathrm{d}t^{M-2}} = \frac{\mathrm{d}^{M-3}u^{\mathrm{m}}(t)}{\mathrm{d}t^{M-3}} = \cdots = \frac{\mathrm{d}u^{\mathrm{m}}(t)}{\mathrm{d}t} = u^{\mathrm{m}}(t) = 0 \tag{41a}$$

$$\frac{\mathrm{d}^{M-1}R_r^{\mathrm{m}}(t)}{\mathrm{d}t^{M-1}} = \frac{\mathrm{d}^{M-2}R_r^{\mathrm{m}}(t)}{\mathrm{d}t^{M-2}} = \cdots = \frac{\mathrm{d}R_r^{\mathrm{m}}(t)}{\mathrm{d}t} = R_r^{\mathrm{m}}(t) = 0 \tag{41b}$$

Equation (32) is transformed to

$$\begin{aligned} &q_0 \underbrace{\int_0^t}_{M \text{ times}} R_r^{\mathrm{m}}(t)\,\mathrm{d}t + q_1 \underbrace{\int_0^t}_{M-1 \text{ times}} R_r^{\mathrm{m}}(t)\,\mathrm{d}t + q_2 \underbrace{\int_0^t}_{M-2 \text{ times}} R_r^{\mathrm{m}}(t)\,\mathrm{d}t + \cdots + R_r^{\mathrm{m}}(t) \\ &= p_0 \underbrace{\int_0^t}_{M \text{ times}} u^{\mathrm{m}}(t)\,\mathrm{d}t + p_1 \underbrace{\int_0^t}_{M-1 \text{ times}} u^{\mathrm{m}}(t)\,\mathrm{d}t + p_2 \underbrace{\int_0^t}_{M-2 \text{ times}} u^{\mathrm{m}}(t)\,\mathrm{d}t + \cdots + p_{M-1} \int_0^t u^{\mathrm{m}}(t)\,\mathrm{d}t \end{aligned} \tag{42}$$

Substituting the Legendre series expansions of $u^m(t)$ (equation (37)) and $R_r^m(t)$ (equation (38)) in equation (42) and using equation (31) leads to

$$\begin{aligned} & q_0\{c_R\}^T[L]^M\{\varphi(t)\} + q_1\{c_R\}^T[L]^{M-1}\{\varphi(t)\} \\ & + q_2\{c_R\}^T[L]^{M-2}\{\varphi(t)\} + \cdots + \{c_R\}^T\{\varphi(t)\} \\ & \cong p_0\{c_u\}^T[L]^M\{\varphi(t)\} + p_1\{c_u\}^T[L]^{M-1}\{\varphi(t)\} \\ & + p_2\{c_u\}^T[L]^{M-2}\{\varphi(t)\} + \cdots + p_{M-1}\{c_u\}^T[L]^T\{\varphi(t)\} \end{aligned} \tag{43}$$

After performing the transpose, the coefficients of $\{\varphi(t)\}^T$ of the left- and right-hand sides are set equal yielding

$$\begin{array}{l} [-([L]^T)^M\{c_R\} \vdots -([L]^T)^{M-1}\{c_R\} \vdots \cdots \vdots -[L]^T\{c_R\} \vdots \\ ([L]^T)^M\{c_u\} \vdots ([L]^T)^{M-1}\{c_u\} \vdots \cdots \vdots [L]^T\{c_u\}] \end{array} \left\{ \begin{array}{c} q_0 \\ q_1 \\ \vdots \\ q_{M-1} \\ p_0 \\ p_1 \\ \vdots \\ p_{M-1} \end{array} \right\} = \{c_R\} \tag{44}$$

This corresponds to making the error defined as the difference between the left- and right-hand sides of equation (43) orthogonal to the l terms of the vector of the function $\{\varphi(t)\}$. The coefficient matrix of equation (44) is of order $l \times 2M$. For a solution $l \geqslant 2M$ must be satisfied. For $l > 2M$ the overdetermined equation is solved using the least-squares procedure yielding the coefficients of the rational approximation $q_0, \ldots, p_{M-1}$ in equation (34).

In a practical application the degree M must be chosen less than M_{max}, as the rank of the coefficient matrix of equation (44) is limited to r_{max}. As the number of unknowns is $2M$, M_{max} is equal to $0{\cdot}5 r_{max}$. An estimation of the upper limit for r_{max} can be established as follows. Each column of the coefficient matrix of equation (44) is a linear combination of the columns of $([L]^T)^i$ ($i = 1, \ldots, M-1$) yielding $([L]^T)^i\{c\}$ where $\{c\}$ is either equal to $\{c_R\}$ or $\{c_u\}$. Using a similarity transformation $[L]^T$ is decomposed as

$$[L]^T = [\Phi][\Lambda][\Phi]^{-1} \tag{45}$$

with the diagonal matrix of the eigenvalues $[\Lambda]$ and the eigenvector matrix $[\Phi]$. This leads to

$$([L]^T)^i = [\Phi][\Lambda]^i[\Phi]^{-1} \tag{46}$$

and

$$([L]^T)^i\{c\} = [\Phi]\{c_i\} \tag{47}$$

where

$$\{c_i\} = [\Lambda]^i[\Phi]^{-1}\{c\} \tag{48}$$

It follows from equation (47) that each column of the coefficient matrix in equation (44) is a linear combination of the columns of $[\Phi]$. An upper limit for r_{max} is thus equal to the rank of $[\Phi]$. $[\Phi]$ is a function of l. The normalized singular values which determine the rank of $[\Phi]$ are plotted for various l in Figure 4. The estimation of the upper limit for r_{max} is based on a condition number (defined as the ratio of the largest to the smallest singular values taken into consideration) equal to 10^4. For a small l (e.g. $=5$), the number of normalized singular values larger than 10^{-4} is equal to l. Increasing l the number of normalized singular values $>10^{-4}$ increases but is bounded by an upper limit of approximately 24 (see $l = 100$). Thus, the numerical rank of the coefficient matrix in equation (44) cannot exceed $r_{max} = 24$. This results in $M_{max} = 12$.

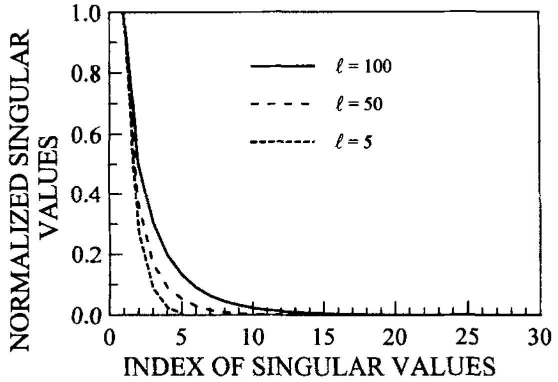

Figure 4. Numerical rank of the eigenvector matrix $[\Phi]$

Three modifications to the procedure to determine the coefficients $q_0, \ldots, p_{M-1}$ are discussed.

In the first procedure the *static-stiffness coefficient* K^{m} is enforced, making the rational approximation doubly asymptotic in the frequency domain. $S^{\mathrm{m}}(\mathrm{i}\omega)$ is decomposed as

$$S^{\mathrm{m}}(\mathrm{i}\omega) = K^{\mathrm{m}}_{\infty} + \mathrm{i}\omega C^{\mathrm{m}}_{\infty} + S^{\mathrm{m}}_r(\mathrm{i}\omega) \tag{49}$$

where $K^{\mathrm{m}}_{\infty} + \mathrm{i}\omega C^{\mathrm{m}}_{\infty}$ represents the singular part. The static-stiffness coefficient K^{m} follows as

$$K^{\mathrm{m}} = S^{\mathrm{m}}(\mathrm{i}\omega = 0) = K^{\mathrm{m}}_{\infty} + S^{\mathrm{m}}_r(\mathrm{i}\omega = 0) \tag{50}$$

Using the Fourier transformation yields

$$K^{\mathrm{m}} = K^{\mathrm{m}}_{\infty} + \int_0^{\infty} S^{\mathrm{m}}_r(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\bigg|_{\mathrm{i}\omega = 0} = K^{\mathrm{m}}_{\infty} + \int_0^{\infty} S^{\mathrm{m}}_r(t)\,\mathrm{d}t \tag{51}$$

Thus, enforcing the static-stiffness coefficient K^{m} with the known K^{m}_{∞} corresponds to equating the integral from zero to ∞ (and not to $t_{\max}$) of the approximate regular part of the unit-impulse response coefficient $\int_0^{\infty} S^{\mathrm{m}}_r(t)\,\mathrm{d}t$ to $K^{\mathrm{m}} - K^{\mathrm{m}}_{\infty}$ which is equal to the corresponding integral of the exact regular part of the unit-impulse response coefficient.

As far as the implementation is concerned, enforcing K^{m} corresponds to equating $S^{\mathrm{m}}_r(\mathrm{i}\omega = 0)$ in equation (34) to $K^{\mathrm{m}} - K^{\mathrm{m}}_{\infty}$. This yields

$$\frac{p_0}{q_0} = K^{\mathrm{m}} - K^{\mathrm{m}}_{\infty} \tag{52}$$

The number of the unknowns is thus reduced by one. This condition can be directly introduced in equation (44) e.g. by eliminating p_0.

In the second procedure the *initial value of the regular part of the unit-impulse response coefficient* $S^{\mathrm{m}}_r(t = 0^+)$ is enforced rigorously. From the initial-value theorem

$$S^{\mathrm{m}}_r(t = 0^+) = \lim_{\omega \to 0} \mathrm{i}\omega S^{\mathrm{m}}_r(\mathrm{i}\omega) \tag{53}$$

and using equation (34) yields

$$p_{M-1} = S^{\mathrm{m}}_r(t = 0^+) \tag{54}$$

In the third procedure, *the regular part of the unit-impulse response coefficient is predistorted* which is achieved by multiplying $S^{\mathrm{m}}_r(t)$ with the factor $\exp[(\beta/t_{\max})t]\,(\beta > 0)$. This places more weight on the approximation of $S^{\mathrm{m}}_r(t)$ for large t ($t \leqslant t_{\max}$) leading to a better approximation for low frequencies. The rational approximation is performed for the product $\exp[(\beta/t_{\max})t]S^{\mathrm{m}}_r(t)$ yielding the coefficients

$q_0^\beta, \ldots, p_{M-1}^\beta$ from solving equation (44). The superscript β is used to denote the quantities associated with the predistorted $\exp[((\beta/t_{\max})/t]S_r^{\mathrm{m}}(t)$. The corresponding partial-fraction expansion of equation (35) is formulated as

$$\frac{p_0^\beta + p_1^\beta(\mathrm{i}\omega) + p_2^\beta(\mathrm{i}\omega)^2 + \cdots + p_{M-1}^\beta(\mathrm{i}\omega)^{M-1}}{q_0^\beta + q_1^\beta(\mathrm{i}\omega) + q_2^\beta(\mathrm{i}\omega)^2 + \cdots + (\mathrm{i}\omega)^M} = \sum_{j=1}^{M} \frac{A_j^\beta}{\mathrm{i}\omega - s_j^\beta} \tag{55}$$

Based on the Laplace transformation it can be shown that the above predistortion in the time domain corresponds to a shift in the Laplace domain by the real value $\beta/t_{\max}$. The rational approximation corresponding to the original $S_r^{\mathrm{m}}(t)$ is obtained by applying the opposite shift yielding

$$\sum_{j=1}^{M} \frac{A_j^\beta}{\mathrm{i}\omega - (s_j^\beta - (\beta/t_{\max}))} = \frac{p_0 + p_1(\mathrm{i}\omega) + p_2(\mathrm{i}\omega)^2 + \cdots + p_{M-1}(\mathrm{i}\omega)^{M-1}}{q_0 + q_1(\mathrm{i}\omega) + q_2(\mathrm{i}\omega)^2 + \cdots + (\mathrm{i}\omega)^M} \tag{56}$$

Equation (56) leads to the coefficients $q_0, \ldots, p_{M-1}$ of the rational approximation.

The rational approximation is performed for the predistorted unit-impulse response coefficient. When returning back to the original unit-impulse response coefficient by multiplying in the time domain by the factor $\exp[-(\beta/t_{\max})t]$, the error in the time domain is multiplied by the same factor. Thus, a reduction of the error for large t results.

In the derivation presented above a set of (shifted) Legendre polynomials are addressed which are defined in the interval $0 \leqslant t \leqslant t_{\max}$ where the regular part of the transformed unit-impulse response coefficient $S_r^{\mathrm{m}}(t)$ is known. Other sets of complete and orthogonal polynomials with respect to a weighting function different from 1 (in the time domain) as the Jacobi polynomials[16] defined in the same interval exist. They are not addressed as the predistortion and the choice of the input to be discussed[11] already permit a weighting effect to be introduced. Complete weighted orthogonal polynomials defined in the interval $0 \leqslant t \leqslant \infty$ such as Laguerre polynomials are not suited as, for $t > t_{\max}$, $S_r^{\mathrm{m}}(t)$ is unknown.

4. STATIC-STIFFNESS, DAMPING AND MASS MATRICES

The singular part in equation (1) contributes directly with the original matrices $[K_\infty]$ and $[C_\infty]$ to the static-stiffness and damping submatrices linking the degrees of freedom on the structure–medium interface.

The regular part is first diagonalized using a transformation without introducing any additional approximation (Section 2) permitting for each element on the diagonal an independent rational approximation to be performed (Section 3). This leads to equation (34). The rational function can be expressed as the partial-fraction expansion of equation (35). If some of the roots s_j are complex, they will appear in complex conjugate pairs $s_{j,j+1} = s_{1j} \pm \mathrm{i}s_{2j}$ whereby the corresponding A_j are also complex conjugate pairs $A_{j,j+1} = A_{1j} \pm \mathrm{i}A_{2j}$. By adding two corresponding first-order terms, a second-order term with real coefficients results. For J conjugate pairs, equation (35) can be written as

$$S_r^{\mathrm{m}}(\mathrm{i}\omega) = \sum_{j=1}^{J} \frac{\beta_{1j}\mathrm{i}\omega + \beta_{0j}}{(\mathrm{i}\omega)^2 + \alpha_{1j}\mathrm{i}\omega + \alpha_{0j}} + \sum_{j=1}^{M-2J} \frac{A_j}{\mathrm{i}\omega - s_j} \tag{57}$$

with

$$\alpha_{0j} = s_{1j}^2 + s_{2j}^2, \qquad \alpha_{1j} = -2s_{1j}, \qquad \beta_{0j} = -2(A_{1j}s_{1j} + A_{2j}s_{2j}), \qquad \beta_{1j} = 2A_{1j} \tag{58a, b, c, d}$$

All coefficients are real.

When the real parts of all the roots s_j are negative, the rational approximation is stable. In this case the corresponding unit-impulse response coefficient decays exponentially with increasing time.

As will be discussed, using a local lumped-parameter model the local (symmetric) static-stiffness and damping matrices and possibly a mass matrix of each element on the diagonal $S_r^{\mathrm{m}}(\mathrm{i}\omega)$ are constructed without introducing any additional approximation. After assemblage of all the local property matrices and backtransformation, the global static-stiffness and damping matrices and possibly a mass matrix are determined (equations (7) or (8)).

The starting point to perform the realization of $S_r^m(i\omega)$ consisting of the local property matrices is the partial fraction expansion (equation (57)) which is exactly in the same form as that derived in Reference 10, where the rational approximation is achieved directly in the frequency domain using the least-squares method applied to $S_r^m(\omega)$. The same procedure can thus be used in the case where $S_r^m(i\omega)$ is constructed via Legendre polynomials in the time domain. To avoid unnecessary duplication only the local property matrices are summarized which are derived using discrete-element models with physical insight. The reader should consult Sections 4.1–4.3 of Reference 10 for details as well as Reference 17.

Each term j of the partial-fraction expansion of the regular part of the dynamic-stiffness coefficient $S_r^m(i\omega)$ (equation (57)) is represented independently from the others. Two possibilities exist leading either to a first-order differential equation (with a static-stiffness matrix and a damping matrix) or a second-order differential equation (with a static-stiffness matrix, a damping matrix and a mass matrix). For the *first-order term* $A_j/(i\omega - s_j)$ the first-order differential equation with the internal variable x_j equals

$$\begin{bmatrix} -A_j/s_j^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x}_j \\ \dot{u}^m \end{Bmatrix} + \begin{bmatrix} A_j/s_j & -A_j/s_j \\ -A_j/s_j & 0 \end{bmatrix} \begin{Bmatrix} x_j \\ u^m \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_{rj}^m \end{Bmatrix} \tag{59}$$

and the second-order differential equation with the internal variable w_j equals

$$\begin{bmatrix} -A_j/s_j^3 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{w}_j \\ \ddot{u}^m \end{Bmatrix} + \begin{bmatrix} A_j/s_j^2 & -A_j/s_j^2 \\ -A_j/s_j^2 & 0 \end{bmatrix} \begin{Bmatrix} \dot{w}_j \\ \dot{u}^m \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -A_j/s_j \end{bmatrix} \begin{Bmatrix} w_j \\ u^m \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_{rj}^m \end{Bmatrix} \tag{60}$$

For the *second-order term* $(\beta_{1j}i\omega + \beta_{0j})/((i\omega)^2 + \alpha_{1j}i\omega + \alpha_{0j})$ the first-order differential equation with the two internal variables x_{1j}, x_{2j} equals

$$\begin{bmatrix} \gamma_{1j} & -\gamma_{1j} & 0 \\ -\gamma_{1j} & \gamma_{1j} + \gamma_{2j} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x}_{1j} \\ \dot{x}_{2j} \\ \dot{u}^m \end{Bmatrix} + \begin{bmatrix} \kappa_{1j} & & -\kappa_{1j} \\ 0 & -\kappa_{2j} & 0 \\ -\kappa_{1j} & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_{1j} \\ x_{2j} \\ u^m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ R_{rj}^m \end{Bmatrix} \tag{61}$$

with

$$\kappa_{1j} = -\frac{\beta_{0j}}{\alpha_{0j}}, \qquad \kappa_{2j} = \frac{\beta_{0j}}{\alpha_{0j}^2} \frac{(-\alpha_{0j}\beta_{1j} + \alpha_{1j}\beta_{0j})^2}{\alpha_{0j}\beta_{1j}^2 - \alpha_{1j}\beta_{0j}\beta_{1j} + \beta_{0j}^2} \tag{62a, b}$$

$$\gamma_{1j} = \frac{\alpha_{0j}\beta_{1j} - \alpha_{1j}\beta_{0j}}{\alpha_{0j}^2}, \qquad \gamma_{2j} = \frac{\beta_{0j}^2}{\alpha_{0j}^2} \frac{-\alpha_{0j}\beta_{1j} + \alpha_{1j}\beta_{0j}}{\alpha_{0j}\beta_{1j}^2 - \alpha_{1j}\beta_{0j}\beta_{1j} + \beta_{0j}^2} \tag{62c, d}$$

and the second-order differential equation with the internal variable w_j equals

$$\begin{bmatrix} \mu_j & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{w}_j \\ \ddot{u}^m \end{Bmatrix} + \begin{bmatrix} 2\gamma_j & -\gamma_j \\ -\gamma_j & 0 \end{bmatrix} \begin{Bmatrix} \dot{w}_j \\ \dot{u}^m \end{Bmatrix} + \begin{bmatrix} 2\kappa_{1j} + \kappa_{2j} & -\kappa_{1j} \\ -\kappa_{1j} & \gamma_j^2/\mu_j \end{bmatrix} \begin{Bmatrix} w_j \\ u^m \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_{rj}^m \end{Bmatrix} \tag{63}$$

μ_j follows from either root of

$$a_j\mu_j^2 + b_j\mu_j + c_j = 0 \tag{64}$$

with

$$a_j = \alpha_{1j}^4 - 4\alpha_{0j}\alpha_{1j}^2, \qquad b_j = -8\alpha_{1j}\beta_{1j} + 16\beta_{0j}, \qquad c_j = 16\frac{\beta_{1j}^2}{\alpha_{1j}^2} \tag{65a, b, c}$$

and

$$\kappa_{1j} = \frac{\mu_j\alpha_{1j}^2}{4} - \frac{\beta_{1j}}{\alpha_{1j}}, \qquad \kappa_{2j} = \mu_j\alpha_{0j} - 2\kappa_{1j}, \qquad \gamma_j = \frac{\mu_j}{2}\alpha_{1j} \tag{66a, b, c}$$

To construct the local property matrices corresponding to the element $S_r^m(i\omega)$, the property matrices of each term of the partial-fraction expansion are assembled. Enforcing compatibility of u^m and equilibrium

$R_r^m = \sum_j R_{rj}^m$ with the sum extended to the $M - 2J$ first-order terms and the J second-order terms as expressed in equation (57) yields for the first-order differential equations

$$\begin{bmatrix} [c_{xx}] & [0] \\ [0] & 0 \end{bmatrix} \begin{Bmatrix} \{\dot{x}^m\} \\ \dot{u}^m \end{Bmatrix} + \begin{bmatrix} [k_{xx}] & [k_{xu}] \\ [k_{xu}]^T & 0 \end{bmatrix} \begin{Bmatrix} \{x^m\} \\ u^m \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ R_r^m \end{Bmatrix} \tag{67}$$

After assembling the local property matrices of all $N(N+1)/2$ diagonal elements, the first-order differential equations corresponding to the diagonal matrix $[S_r^m(i\omega)]$ are formulated as

$$\begin{bmatrix} [C_{xx}] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\dot{x}\} \\ \{\dot{u}^m\} \end{Bmatrix} + \begin{bmatrix} [K_{xx}] & [K_{xu}] \\ [K_{xu}]^T & [0] \end{bmatrix} \begin{Bmatrix} \{x\} \\ \{u^m\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{R_r^m\} \end{Bmatrix} \tag{68}$$

Performing the backtransformation by substituting equation (11) in equation (68) leads to the global property matrices corresponding to the regular part. After assembling the property matrices $[K_\infty]$ and $[C_\infty]$ of the singular part, the first-order differential equations corresponding to the rational approximation of the unbounded medium result

$$\begin{bmatrix} [C_{xx}] & [0] \\ [0] & [C_\infty] \end{bmatrix} \begin{Bmatrix} \{\dot{x}\} \\ \{\dot{u}\} \end{Bmatrix} + \begin{bmatrix} [K_{xx}] & [K_{xu}][T]^T \\ [T][K_{xu}]^T & [K_\infty] \end{bmatrix} \begin{Bmatrix} \{x\} \\ \{u\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{R\} \end{Bmatrix} \tag{69}$$

Equation (69) is the same relation as equation (7) which identifies $[C^1]$ and $[K^1]$.

Proceeding in the same manner for the second-order differential equations yields

$$\begin{bmatrix} [M_{ww}] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{w}\} \\ \{\ddot{u}\} \end{Bmatrix} + \begin{bmatrix} [C_{ww}] & [C_{wu}][T]^T \\ [T][C_{wu}]^T & [C_\infty] \end{bmatrix} \begin{Bmatrix} \{\dot{w}\} \\ \{\dot{u}\} \end{Bmatrix}$$
$$+ \begin{bmatrix} [K_{ww}] & [K_{wu}][T]^T \\ [T][K_{wu}]^T & [T][K_{uu}][T]^T + [K_\infty] \end{bmatrix} \begin{Bmatrix} \{w\} \\ \{u\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{R\} \end{Bmatrix} \tag{70}$$

which is the same relation as equation (8) defining $[M^2]$, $[C^2]$ and $[K^2]$.

5. CONCLUDING REMARKS

1. A systematic procedure to construct the (symmetric) static-stiffness, damping and mass matrices representing the unbounded medium is presented addressing the unit-impulse response matrix in the time domain corresponding to the degrees of freedom on the structure–medium interface. The unit-impulse response matrix is first diagonalized without introducing any approximation which then allows each term to be modelled independently from the others. The suitably chosen input and the corresponding output are expanded in a series of Legendre polynomials in the time domain which permits the rational approximation in the frequency domain of the dynamic-stiffness coefficient to be determined by solving a linear system of equations only. In addition, the static-stiffness coefficient can be enforced which results in a doubly-asymptotic procedure as well as the initial value of the regular part of the unit-impulse response coefficient. The unit-impulse response coefficient can also be predistorted.
2. Using a lumped-parameter model which provides physical insight the static-stiffness, damping and mass matrices are constructed without introducing any additional approximation.
3. The present procedure using the unit-impulse response matrix in the time domain with Legendre polynomials is analogous to the least-squares method addressing the dynamic-stiffness matrix in the frequency domain, both yielding a rational approximation in the frequency domain.
4. The unbounded medium is modelled in the same manner as the structure and thus the same computer programme can be used for dynamic unbounded medium–structure-interaction analysis as for structural dynamics.
5. The implementation and stringent tests consisting of dispersive systems with a cutoff frequency are discussed in the accompanying paper.[11]

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